COMS 4995: Introduction to Communication Complexity — Sep 12, 2011

Lecture 2 – Fooling Sets, Rectangle Size and Rank Lower Bound

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1 Lower Bound Techniques

In this lecture we discuss three techniques to lower bound the deterministic communication complexity of a function. All methods are based on the following combinatorial observation (check previous notes for a proof).

Lemma 1. *If any partition of $X \times Y$ into f-monochromatic rectangles requires at least k rectangles, then $D(f) \geq \lceil \log_2 k \rceil$.*

Hence to lower bound the deterministic communication complexity of f it is sufficient to lower bound the number of rectangles in any partition of $X \times Y$ into f-monochromatic rectangles.

1.1 Fooling Sets

Roughly speaking, a fooling set is a subset of the domain of f such that no two elements of it can lie in the same f-monochromatic rectangle. Therefore, if we can prove the existence of a fooling set of size k for f then every partition of $X \times Y$ into f-monochromatic rectangles contains at least k rectangles. Formally:

Definition 2. *Let $f : X \times Y \to \{0, 1\}$. A set $S \subseteq X \times Y$ is called a* fooling set *if there exists a value $b \in \{0, 1\}$ such that*

- *For every $(x, y) \in S$, $f(x, y) = b$.*
- *For every two distinct pairs (x_1, y_1) and (x_2, y_2) in S, either $f(x_1, y_2) \neq b$ or $f(x_2, y_1) \neq b$.*

Lemma 3. *If f admits a fooling set S of size k, then $D(f) \geq \lceil \log_2 k \rceil$.*

Proof. Remember that if R is a rectangle then $(x_1, y_1) \in R$ and $(x_2, y_2) \in R$ imply that $(x_1, y_2) \in R$ and $(x_2, y_1) \in R$. It follows by the definition of a fooling set that no two distinct elements of S can lie in the same f-monochromatic rectangle. Hence every partition of M_f into f-monochromatic rectangles contains at least k rectangles. It follows from lemma 1 that $D(f) \geq \lceil \log_2 k \rceil$. □

Next we provide a few examples for which the fooling set method turns out to be useful.

Example 4. *The equality function $EQ_n(x, y) : \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$ is defined to be 1 if and only if $x = y$. It is easy to check that a fooling set of size 2^n for this function is*

$$S = \{(z, z) \mid z \in \{0, 1\}^n\}.$$

It follows from lemma 3 that $D(EQ_n) \geq n$.

Example 5. *The disjointness function $DISJ_n(x, y) : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is defined to be 1 if and only if vectors x and y when interpreted as subsets of $[n]$ are disjoint. It is not difficult to check that a fooling set of size 2^n for this function is*

$$S = \{(A, \bar{A}) \mid A \subseteq [n]\}.$$

It follows from lemma 3 that $D(DISJ_n) \geq n$.

We note that counting 0-rectangles it is possible to improve these bounds to $D(\mathrm{EQ}_n) \geq n+1$ and $D(\mathrm{DISJ}_n) \geq n+1$.

1.2 Rectangle Size

This method consists of upper bounding the maximum size $t = |A||B|$ of a $R = A \times B$ f-monochromatic rectangle. If f is a function on $X \times Y$, then it is clear that the minimum number of rectangles in any partition of $X \times Y$ into f-monochromatic rectangles will be $|X||Y|/t$. We can also upper bound the size of 0-monochromatic rectangles and lower bound the number of input pairs (x, y) for which $f(x, y) = 0$.

Example 6. *The inner product function $IP_n(x, y) : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is defined by $IP_n(x, y) = \langle x, y \rangle$ over $\mathbb{F}_2$. First note that the number of input pairs (x, y) for which $IP_n(x, y) = 0$ is exactly $2^n + (2^n - 1)2^n/2$ (consider separately the 0^n input and the rest of the inputs and count the number of zeroes in each line of the matrix M_{IP_n}). Next we prove that the size of any 0-monochromatic rectangle for this function is bounded by 2^n. It follows that in any partition of $X \times Y$ into f-monochromatic rectangles there are at least $\lceil 2^{n-1} + 1/2 \rceil$ 0-rectangles. Thus by lemma 1 we get $D(IP_n) \geq n$.*

Let $R = A \times B$ be a 0-rectangle. To upper bound the size of R we use the linear algebra method. Let $A' = span(A)$ and $B' = span(B)$ (over $\mathbb{F}_2$). Note that A' and B' may have larger area but $A' \times B'$ is still a 0-monochromatic rectangle, since

$$\langle a + a', b + b' \rangle = \langle a, b \rangle + \langle a, b' \rangle + \langle a', b \rangle + \langle a', b' \rangle = 0.$$

Now since A' and B' are orthogonal subspaces of $\mathbb{F}_2^n$, it follows by linear algebra that the sum of $dim(A')$ and $dim(B')$ is at most n (the dimension of $\mathbb{F}_2^n$). Hence

$$|A||B| \leq |A'||B'| = 2^{dim(A')} 2^{dim(B')} \leq 2^n.$$

The rectangle size method and the fooling set method can be interpreted as particular cases of the following general observation.

Lemma 7. *Let μ be a probability distribution of $X \times Y$. If any f-monochromatic rectangle R has measure $\mu(R) \leq \delta$, then $D(f) \geq \lceil \log_2 1/\delta \rceil$.*

Proof. Since μ is a probability distribution we get $\mu(X \times Y) = 1$. Hence there must be at least $1/\delta$ rectangles in any partition of f into f-monochromatic rectangles. The result then follows from lemma 1. □

1.3 Rank

This technique uses an algebraic method to lower bound the number of rectangles in any f-monochromatic partition of $X \times Y$.

Definition 8. *$rank(f)$ is the rank of the matrix M_f over $\mathbb{R}$.*

Lemma 9. *For any function $f : X \times Y \to \{0, 1\}$,*

$$D(f) \geq \lceil \log_2 rank(M_f) \rceil.$$

Proof. By lemma 1, it is enough to show that the rank of M_f is a lower bound on the number of rectangles in any f-monochromatic partition of $X \times Y$. We actually prove the stronger result that $rank(M_f)$ is a lower bound on the number of 1-rectangles in any such partition. To see this, given a f-monochromatic partition $\mathcal{P}$ of $X \times Y$, where $R_1, R_2, \ldots, R_k$ are 1-monochromatic rectangles, write $M_f = M_1 + M_2 + \ldots + M_k$, where each M_i is the natural 0/1-incidence matrix related to R_i. Now apply the subadditivity of the rank function to get $rank(M_f) \leq rank(M_1) + rank(M_2) + \ldots + rank(M_k) = k \leq |\mathcal{P}|$. □

Example 10. *The equality function EQ_n has rank 2^n, hence by lemma 9 we have $D(EQ_n) \geq n$.*

Example 11. *Using the rank method we can give an alternative proof that the inner product function IP_n requires n bits of communication. Let M be the 0/1-matrix representing this function. If we use the fact that $rank(M) \geq rank(M^2)$ then it is enough to lower bound the rank of M^2, which appears to be an easier task. Note that the entry of M^2 indexed by (x, y) is given by $\sum_{z \in \{0,1\}^n} \langle x, z\rangle\langle z, y\rangle$, i.e., it is exactly the number of z's such that $\langle x, z\rangle = \langle z, y\rangle = 1$.*

We will actually lower bound the rank of the submatrix M' of M^2 obtained from M^2 by deleting its first row. Let $x \neq 0$ and i be some non-zero coordinate of x. Then for any z, exactly one of the equations $\langle x, z\rangle = 1$ and $\langle x, z^{(i)}\rangle = 1$ is true, where $z^{(i)}$ is obtained from z by flipping its i-th coordinate. It follows that for $x = y$ we have $M'_{(x,y)} = 2^{n-1}$.

For $x \neq y$ both non-zero we can use the fact that the lines M_x and M_y of the matrix M are lines of a Hadamard matrix of order 2^n (after a convenient bitwise transformation given by $b \to (-1)^b$) to conclude that exactly a 1/4 fraction of the inputs z satisfy $\langle x, z\rangle = \langle z, y\rangle = 1$. Therefore for $x \neq y$ we have $M'_{(x,y)} = 2^{n-2}$.

It follows that M' is a diagonal matrix with the value 2^{n-1} on the main diagonal and the value 2^{n-2} off the diagonal. Therefore $rank(M') = 2^n - 1$, i.e., the number of rows in M'. It follows that $rank(M) \geq rank(M^2) \geq rank(M') = 2^n - 1$. By lemma 9, $D(IP_n) \geq n$.