COMS 4995: Introduction to Communication Complexity — Sep 14, 2011

Lecture 3 – Covers and Nondeterminism

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1 Basic Measures in Communication Complexity

To investigate the power of the lower bound methods discussed in the previous lectures it will be convenient to define the following measures.

Definition 1. *Let $f : X \times Y \to \{0,1\}$ be a function.*

- *$D(f)$ is the deterministic communication complexity of f.*
- *$C^P(f)$ is the smallest number of leaves in a protocol tree for f.*
- *$C^D(f)$ is the smallest number of f-monochromatic rectangles in a partition of $X \times Y$.*
- *$C(f)$ is the smallest number of f-monochromatic rectangles needed to cover $X \times Y$.*
- *$C^0(f)$ is the smallest number of rectangles needed to cover the 0-inputs of f.*
- *$C^1(f)$ is the smallest number of rectangles needed to cover the 1-inputs of f.*

First we state some elementary inequalities involving these measures.

Lemma 2. *For any function $f : X \times Y \to \{0,1\}$ it holds that*

$$C^0(f) + C^1(f) = C(f) \le C^D(f) \le C^P(f) \le 2^{D(f)}.$$

Note that the fooling set method and the rectangle size method actually provide a lower bound on $C(f)$. On the other hand, the rank method only gives us a lower bound on the partition number $C^D(f)$ of f.

Example 3. *We have $C^1(\overline{\mathrm{EQ}}) \le 2n$, since we can cover the 1-inputs of $\overline{\mathrm{EQ}}$ taking for each $i \in [n]$ the 1-rectangles $\{x \mid x_i = 1\} \times \{y \mid y_i = 0\}$ and $\{x \mid x_i = 0\} \times \{y \mid y_i = 1\}$.*

2 Nondeterminism

In this section we define a nondeterministic model of communication which provides the parties access to a public advice string $a \in \{0,1\}^L$ in addition to their respective private inputs x and y. The structure of the new model is essentially the same used before with the exception that at each internal node of the protocol tree we have now a function that depends both on the private input (x or y) and on the public input a. We write $P_a(x,y)$ for the output of the protocol when executed with advice string a. Finally, we say that P is a nondeterministic protocol for f using advice string of size L if the following conditions are satisfied:

- If $f(x, y) = 1$ then $P_a(x, y) = 1$ for some advice string $a \in \{0, 1\}^L$.
- If $f(x, y) = 0$ then $P_a(x, y) = 0$ for every advice string $a \in \{0, 1\}^L$.

The cost of a nondeterministic protocol P that uses advice strings of length L is defined to be $\mathsf{height}(P)+L$. Given a function f, its nondeterministic complexity $N(f)$ is the smallest cost among the nondeterministic protocols for f.

Lemma 4. *For any function f:*

$$\lceil \log_2 C^1(f) \rceil \leq N(f) \leq \lceil \log_2 C^1(f) \rceil + 2.$$

Proof. First we argue that $N(f) \leq \lceil \log_2 C^1(f) \rceil + 2$. We need to exhibit a nondeterministic protocol P such that $\mathsf{cost}(P) \leq \lceil \log_2 C^1(f) \rceil + 2$. Alice and Bob have access to a public string $a \in \{0, 1\}^{\lceil \log_2 C^1(f) \rceil}$ that is interpreted as the index of a 1-rectangle R (for some fixed cover of the 1-inputs of size $C^1(f)$). First Bob informs Alice (using one bit) whether $x \in R$. Then Alice outputs 1 if and only if both $x \in R$ and $y \in R$. It is easy to check that this is a correct nondeterministic protocol for f. The total cost of the protocol is exactly $\lceil \log_2 C^1(f) \rceil + 2$, as required.

Now we move to the proof of the first inequality. Let P be a nondeterministic protocol for f with $t = \mathsf{cost}(P) = \mathsf{height}(P) + L$, where L is the size of the advice string. Since $N(f)$ is an integer function, it is enough to show that $C^1(f) \leq 2^t$. Let S_a be the set of 1-leaves of the protocol tree given by P_a, where a is some advice string of length L. Clearly, $|S_a| \leq \mathsf{height}(P)$. Also, there are no more than 2^L sets S_a. Now for each advice string a and for each 1-leave $l \in S_a$, let $R_{a,l}$ be the corresponding 1-rectangle associated to l (note that it follows from the second property of the definition of nondeterministic protocol that the rectangle is indeed a 1-rectangle). In addition, it is easy to see that these $2^L \times 2^{\mathsf{height}(P)} = 2^t$ rectangles constitute a 1-cover of f.

□

Therefore to determine the nondeterministic communication complexity of f it is enough to study the number of monochromatic rectangles needed to cover the 1-inputs of f.

3 Determinism Versus Nondeterminism

In this section we prove a few fundamental results relating the deterministic and nondeterministic communication complexity of a function.

Theorem 5. $\log_2(D(f) - 1) \leq N(f)$.

Proof. The result follows from exercise 2.6 of the textbook (which is incidentally one of the homework problems) and lemma 4. □

Note that this bound is tight since it is not difficult to come up with a $O(\log n)$ nondeterministic protocol for the funcion $\overline{\mathsf{EQ}_n}$.

Theorem 6. $D(f) \leq O(N(f)N(\overline{f}))$.

Proof. By lemma 4, it is enough to show that $D(f) = O(\log C^1(f) \log C^0(f))$. We provide an algorithmic proof that is based on the following simple observation: if $R = S \times T$ is a 0-monochromatic rectangle and $R' = S' \times T'$ is a 1-monochromatic rectangle, then either $S \cap S' = \emptyset$ or $T \cap T' = \emptyset$.

Fix a cover of the 0-inputs of size $C^0(f)$ and a cover of the 1-inputs of size $C^1(f)$. Alice and Bob search for a 1-rectangle containing the input. If there is no such rectangle, they conclude that $f(x, y) = 0$. In each stage their reduce the number of 1-rectangles in the search space by at least a factor of two, while exchanging at most $O(\log C^0(f))$ bits per stage. Hence the total communication complexity is $O(\log C^1(f) \log C^0(f))$, as desired.

Initially all 1-rectangles belong to the search space, and throughout the protocol the players will consistently update the search space in their minds. In each stage they proceed as follows:

> 1) Alice's turn. If the search space is empty then Alice outputs $f(x, y) = 0$. Otherwise, Alice tries to find a 0-rectangle R that contains the row of x and that intersects in rows with no more than half of the remaining 1-rectangles. If Alice succeeds, she sends R to Bob and both players update the search space (the 1-rectangles that remain are those that intersect in rows with R). If there is no such rectangle R, then Alice tells Bob that this is the case.
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> 2) Bob's turn. Bob updates the search space using the message received from Alice. If the search space is empty then Bob outputs $f(x, y) = 0$. Otherwise, Bob looks for a 0-rectangle R that contains the column y and that intersects in columns with no more than half of the 1-rectangles remaining in the search space. If such R exists, Bob sends it to Alice and both players update the search space. A new stage begins if at least one of the players succeeded in finding a good 0-rectangle. Otherwise, if both players fail, Bob outputs $f(x, y) = 1$.

It remains to prove the correctness of the protocol. If either Alice or Bob outputs $f(x, y) = 0$ then the protocol is correct, since then it must be the case that there is no remaining 1-rectangle in the search space (note that if (x, y) is in a 1-rectangle Q then Q remains alive during the protocol). On the other hand, Bob only outputs $f(x, y) = 1$ when they both fail in finding a good 0-rectangle R. But in this case we must have $f(x, y) = 1$. This is the case because if $f(x, y) = 0$ then (x, y) must lie in some 0-rectangle, and then by the simple observation stated before this 0-rectangle cannot intersect in rows more than half of the remaining 1-rectangles and intersect in columns more than half of the remaining 1-rectangles at the same time (otherwise it would intersect in row and in column some 1-rectangle). □

Corollary 7. $D(f) \leq O((\log C^D(f))^2)$.

Proof. By lemma 4, we have $N(f) \leq \log_2 C^1(f) + 2 \leq \log_2 C^D(f) + 2$ and $N(\overline{f}) \leq \log_2 C^0(f) + 2 \leq \log_2 C^D(f) + 2$. The result follows then from theorem 6. □

4 Additional Inequalities

We comment on a few additional results discussed in class. The next lemma shows that the protocol partition number $C^P(f)$ of f essentially characterizes its (deterministic) communication complexity.

Lemma 8. $\log_2 C^P(f) \leq D(f) \leq O(\log C^P(f))$.

Finally, it is possible to prove that the probability distribution method (the generalization of the rectangle method using a distribution μ over $X \times Y$) can provide close to optimal lower bounds on the nondeterministic communication complexity of a funcion, i.e., it is always possible to define a convenient probability distribution μ over $X \times Y$ that would give us a good lower bound on $C^1(f)$. On the other hand, it is possible to prove that the fooling set method does not always suffice. See the textbook for more details.