

## Lecture 3 – Covers and Nondeterminism

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## 1 Basic Measures in Communication Complexity

To investigate the power of the lower bound methods discussed in the previous lectures it will be convenient to define the following measures.

**Definition 1.** Let  $f : X \times Y \rightarrow \{0, 1\}$  be a function.

- $D(f)$  is the deterministic communication complexity of  $f$ .
- $C^P(f)$  is the smallest number of leaves in a protocol tree for  $f$ .
- $C^D(f)$  is the smallest number of  $f$ -monochromatic rectangles in a partition of  $X \times Y$ .
- $C(f)$  is the smallest number of  $f$ -monochromatic rectangles needed to cover  $X \times Y$ .
- $C^0(f)$  is the smallest number of rectangles needed to cover the 0-inputs of  $f$ .
- $C^1(f)$  is the smallest number of rectangles needed to cover the 1-inputs of  $f$ .

First we state some elementary inequalities involving these measures.

**Lemma 2.** For any function  $f : X \times Y \rightarrow \{0, 1\}$  it holds that

$$C^0(f) + C^1(f) = C(f) \leq C^D(f) \leq C^P(f) \leq 2^{D(f)}.$$

Note that the fooling set method and the rectangle size method actually provide a lower bound on  $C(f)$ . On the other hand, the rank method only gives us a lower bound on the partition number  $C^D(f)$  of  $f$ .

**Example 3.** We have  $C^1(\overline{\text{EQ}}) \leq 2n$ , since we can cover the 1-inputs of  $\overline{\text{EQ}}$  taking for each  $i \in [n]$  the 1-rectangles  $\{x \mid x_i = 1\} \times \{y \mid y_i = 0\}$  and  $\{x \mid x_i = 0\} \times \{y \mid y_i = 1\}$ .

## 2 Nondeterminism

In this section we define a nondeterministic model of communication which provides the parties access to a public advice string  $a \in \{0, 1\}^L$  in addition to their respective private inputs  $x$  and  $y$ . The structure of the new model is essentially the same used before with the exception that at each internal node of the protocol tree we have now a function that depends both on the private input ( $x$  or  $y$ ) and on the public input  $a$ . We write  $P_a(x, y)$  for the output of the protocol when executed with advice string  $a$ . Finally, we say that  $P$  is a nondeterministic protocol for  $f$  using advice string of size  $L$  if the following conditions are satisfied:

- If  $f(x, y) = 1$  then  $P_a(x, y) = 1$  for some advice string  $a \in \{0, 1\}^L$ .
- If  $f(x, y) = 0$  then  $P_a(x, y) = 0$  for every advice string  $a \in \{0, 1\}^L$ .

The cost of a nondeterministic protocol  $P$  that uses advice strings of length  $L$  is defined to be  $\text{height}(P) + L$ . Given a function  $f$ , its nondeterministic complexity  $N(f)$  is the smallest cost among the nondeterministic protocols for  $f$ .

**Lemma 4.** *For any function  $f$ :*

$$\lceil \log_2 C^1(f) \rceil \leq N(f) \leq \lceil \log_2 C^1(f) \rceil + 2.$$

*Proof.* First we argue that  $N(f) \leq \lceil \log_2 C^1(f) \rceil + 2$ . We need to exhibit a nondeterministic protocol  $P$  such that  $\text{cost}(P) \leq \lceil \log_2 C^1(f) \rceil + 2$ . Alice and Bob have access to a public string  $a \in \{0, 1\}^{\lceil \log_2 C^1(f) \rceil}$  that is interpreted as the index of a 1-rectangle  $R$  (for some fixed cover of the 1-inputs of size  $C^1(f)$ ). First Bob informs Alice (using one bit) whether  $x \in R$ . Then Alice outputs 1 if and only if both  $x \in R$  and  $y \in R$ . It is easy to check that this is a correct nondeterministic protocol for  $f$ . The total cost of the protocol is exactly  $\lceil \log_2 C^1(f) \rceil + 2$ , as required.

Now we move to the proof of the first inequality. Let  $P$  be a nondeterministic protocol for  $f$  with  $t = \text{cost}(P) = \text{height}(P) + L$ , where  $L$  is the size of the advice string. Since  $N(f)$  is an integer function, it is enough to show that  $C^1(f) \leq 2^t$ . Let  $S_a$  be the set of 1-leaves of the protocol tree given by  $P_a$ , where  $a$  is some advice string of length  $L$ . Clearly,  $|S_a| \leq \text{height}(P)$ . Also, there are no more than  $2^L$  sets  $S_a$ . Now for each advice string  $a$  and for each 1-leave  $l \in S_a$ , let  $R_{a,l}$  be the corresponding 1-rectangle associated to  $l$  (note that it follows from the second property of the definition of nondeterministic protocol that the rectangle is indeed a 1-rectangle). In addition, it is easy to see that these  $2^L \times 2^{\text{height}(P)} = 2^t$  rectangles constitute a 1-cover of  $f$ . □

Therefore to determine the nondeterministic communication complexity of  $f$  it is enough to study the number of monochromatic rectangles needed to cover the 1-inputs of  $f$ .

### 3 Determinism Versus Nondeterminism

In this section we prove a few fundamental results relating the deterministic and nondeterministic communication complexity of a function.

**Theorem 5.**  $\log_2(D(f) - 1) \leq N(f)$ .

*Proof.* The result follows from exercise 2.6 of the textbook (which is incidentally one of the homework problems) and lemma 4. □

Note that this bound is tight since it is not difficult to come up with a  $O(\log n)$  nondeterministic protocol for the function  $\overline{\text{EQ}}_n$ .

**Theorem 6.**  $D(f) \leq O(N(f)N(\overline{f}))$ .

*Proof.* By lemma 4, it is enough to show that  $D(f) = O(\log C^1(f) \log C^0(f))$ . We provide an algorithmic proof that is based on the following simple observation: if  $R = S \times T$  is a 0-monochromatic rectangle and  $R' = S' \times T'$  is a 1-monochromatic rectangle, then either  $S \cap S' = \emptyset$  or  $T \cap T' = \emptyset$ .

Fix a cover of the 0-inputs of size  $C^0(f)$  and a cover of the 1-inputs of size  $C^1(f)$ . Alice and Bob search for a 1-rectangle containing the input. If there is no such rectangle, they conclude that  $f(x, y) = 0$ . In each stage they reduce the number of 1-rectangles in the search space by at least a factor of two, while exchanging at most  $O(\log C^0(f))$  bits per stage. Hence the total communication complexity is  $O(\log C^1(f) \log C^0(f))$ , as desired.

Initially all 1-rectangles belong to the search space, and throughout the protocol the players will consistently update the search space in their minds. In each stage they proceed as follows:

1) Alice's turn. If the search space is empty then Alice outputs  $f(x, y) = 0$ . Otherwise, Alice tries to find a 0-rectangle  $R$  that contains the row of  $x$  and that intersects in rows with no more than half of the remaining 1-rectangles. If Alice succeeds, she sends  $R$  to Bob and both players update the search space (the 1-rectangles that remain are those that intersect in rows with  $R$ ). If there is no such rectangle  $R$ , then Alice tells Bob that this is the case.

2) Bob's turn. Bob updates the search space using the message received from Alice. If the search space is empty then Bob outputs  $f(x, y) = 0$ . Otherwise, Bob looks for a 0-rectangle  $R$  that contains the column  $y$  and that intersects in columns with no more than half of the 1-rectangles remaining in the search space. If such  $R$  exists, Bob sends it to Alice and both players update the search space. A new stage begins if at least one of the players succeeded in finding a good 0-rectangle. Otherwise, if both players fail, Bob outputs  $f(x, y) = 1$ .

It remains to prove the correctness of the protocol. If either Alice or Bob outputs  $f(x, y) = 0$  then the protocol is correct, since then it must be the case that there is no remaining 1-rectangle in the search space (note that if  $(x, y)$  is in a 1-rectangle  $Q$  then  $Q$  remains alive during the protocol). On the other hand, Bob only outputs  $f(x, y) = 1$  when they both fail in finding a good 0-rectangle  $R$ . But in this case we must have  $f(x, y) = 1$ . This is the case because if  $f(x, y) = 0$  then  $(x, y)$  must lie in some 0-rectangle, and then by the simple observation stated before this 0-rectangle cannot intersect in rows more than half of the remaining 1-rectangles and intersect in columns more than half of the remaining 1-rectangles at the same time (otherwise it would intersect in row and in column some 1-rectangle). □

**Corollary 7.**  $D(f) \leq O((\log C^D(f))^2)$ .

*Proof.* By lemma 4, we have  $N(f) \leq \log_2 C^1(f) + 2 \leq \log_2 C^D(f) + 2$  and  $N(\bar{f}) \leq \log_2 C^0(f) + 2 \leq \log_2 C^D(f) + 2$ . The result follows then from theorem 6. □

## 4 Additional Inequalities

We comment on a few additional results discussed in class. The next lemma shows that the protocol partition number  $C^P(f)$  of  $f$  essentially characterizes its (deterministic) communication complexity.

**Lemma 8.**  $\log_2 C^P(f) \leq D(f) \leq O(\log C^P(f))$ .

Finally, it is possible to prove that the probability distribution method (the generalization of the rectangle method using a distribution  $\mu$  over  $X \times Y$ ) can provide close to optimal lower bounds on the nondeterministic communication complexity of a function, i.e., it is always possible to define a convenient probability distribution  $\mu$  over  $X \times Y$  that would give us a good lower bound on  $C^1(f)$ . On the other hand, it is possible to prove that the fooling set method does not always suffice. See the textbook for more details.